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Dimensional reduction in supersymmetric field theories

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Abstract

A class of quantum field theories invariant with respect to the action of an odd vector field Q on a source supermanifold Σ is considered. We suppose that Q satisfies the conditions of one of the localization theorems (Szabo R 1996 Equivariant localization of path integrals *Preprint* hep-th/9608068). The Q-invariant sector of a field theory from the above class is then shown to be equivalent to the quantum field theory defined on the zero locus of the vector field Q.

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1. Introduction

The aim of the present paper is to connect the phenomenon of dimensional reduction in supersymmetric field theories with localization of certain integrals over supermanifolds. Let us start with the explanation of the meaning we assign to the terms 'dimensional reduction' and 'localization' (each of these terms is used in the literature in quite a few different contexts). By dimensional reduction, we will understand the fact of an *exact* equivalence between a quantum field theory and another quantum field theory defined on the submanifold of the source manifold of the original theory. An example of such a phenomenon is provided by the celebrated Parisi–Sourlas model [5] which also served as a main motivation for the present work. Recently, the Parisi–Sourlas method has been used to compute exact scaling functions in certain classical statistical systems, namely self-avoiding walks and branched polymers [3]. The generalization of the Parisi–Sourlas dimensional reduction argument presented below might prove useful in the similar study of constrained classical statistical systems.

Localization of an integral over a (super)manifold Σ to a subset $R \subset \Sigma$ means more or less that this integral is independent of the values of the integrand on the complement to the arbitrary neighbourhood of R in Σ. In what follows, we will be using the notion of localization in the even more restricted sense. It is well known that localization is usually related to the presence of some odd symmetry of the problem. So let Q be an odd vector field on Σ. This means that Q is a parity-reversing derivation on the Z_2-graded algebra of functions on Σ. We say that Q satisfies the conditions of some localization theorem if for any Q-invariant function f on Σ,

$$\int_\Sigma \mathrm{d}V \cdot f = \int_{R_Q} \mathrm{d}v_Q f|_{R_Q} \tag{1}$$

where $\mathrm{d}V$ is a fixed volume element on Σ; the zero locus of Q is supposed to be a submanifold of Σ and is denoted by R_Q; $\mathrm{d}v_Q$ stands for the volume element on R_Q depending on $\mathrm{d}V$, Q, but not f.

An exposition of different localization techniques in the context of quantum field theory can be found in [11]. Most of the localization theorems discussed in this paper (such as the Duistermaat–Heckmann theorem, Berline–Vergne theorem, Mathai–Quillen formula) state localization of integrals over supermanifolds under certain compactness conditions imposed on the anticommutator $\{Q, Q\}$. In [10] we studied such localization statements in the framework of supergeometry. We managed to prove a general localization theorem which includes all localization theorems mentioned above as its particular cases. The main result of [10] can be formulated as follows. Let Q be an odd vector field on Σ which preserves a volume element $\mathrm{d}V$ on Σ. Suppose that $Q^2 = \frac{1}{2}\{Q, Q\}$ belongs to a Lie algebra of a compact subgroup of the group of diffeomorphisms of Σ. Then under some additional conditions of non-degeneracy of Q the integrals of Q-invariant functions over Σ localize to the zero locus R_Q of the vector field Q. In other words (1) holds with $\mathrm{d}v_Q$ determined by $\mathrm{d}V$ and the matrix of the first derivatives of the vector field Q at R_Q.

To conclude the introduction, let us formulate and prove another localization theorem which will be useful in the analysis of dimensional reduction of Parisi–Sourlas-type models. This theorem can be viewed as a supersymmetric version of the localization formula for the functional of isometry generators, see [11], paragraph 4.6. The explanation of basic notions of supergeometry which will be used below can be found in [9].

Theorem. *Let Σ be a compact supermanifold equipped with an even metric g. Suppose Q is an odd vector field on Σ preserving the metric, i.e. $L_Q g = 0$. Suppose that vector field Q^2 is non-degenerate in the vicinity of its zero locus R_{Q^2}. Suppose also that odd and even codimensions of R_{Q^2} in Σ coincide. Then for any Q-invariant function f on Σ*

$$\int_\Sigma \mathrm{d}V f = \int_{R_{Q^2}} \mathrm{d}v_Q f|_{R_{Q^2}} \tag{2}$$

where $\mathrm{d}V$ is a volume element on Σ corresponding to the metric g and $\mathrm{d}v_Q$ is a volume element on R_{Q^2} determined completely by g and Q.[1]

Proof. Let $\{z^\alpha\}$ be a set of local coordinates on Σ. The parity of the αth coordinate will be denoted by ϵ_α. In these coordinates $Q = Q^\alpha(z)\frac{\partial}{\partial z^\alpha}$, $Q^2 = (Q^2)^\alpha(z)\frac{\partial}{\partial z^\alpha}$, where $(Q^2)^\alpha = Q(Q^\alpha(z))$. We will write the metric in the form $g = g_{\alpha\beta}(z)\delta z^\alpha \delta z^\beta$. Consider now an odd function σ on Σ defined in the local coordinates by the following expression:

$$\sigma(z) = \frac{1}{2}\sum_{\alpha,\beta}(-1)^{\epsilon_\alpha+\epsilon_\beta} g_{\alpha\beta} Q^\alpha(z)(Q^2)^\beta(z). \tag{3}$$

It is easy to verify that the right-hand side of (3) does not depend on the choice of local coordinates, so that σ is indeed a function on Σ. A direct calculation shows that

[1] Initially this theorem was formulated and proved for the linear superspaces. Its present form benefits from the collaboration with A S Schwarz.

σ is Q^2-invariant, i.e. $Q^2\sigma = 0$. Also one finds that $Q\sigma(z) = g_{\alpha\beta}(Q^2)^\alpha(z)(Q^2)^\beta(z) \equiv \langle Q^2(z), Q^2(z)\rangle$, where $\langle\ ,\ \rangle$ denotes the pairing in the fibres of the tangent bundle over Σ induced by the metric g.

Another computation shows that R_{Q^2} is a subset of the critical set of $Q\sigma$, i.e. $\nabla Q\sigma|_{R_{Q^2}} = 0$. It follows from non-degeneracy of Q^2 in the vicinity of R_{Q^2} that R_{Q^2} is a non-degenerate critical set. The last means that the Hessian of $Q\sigma$ has the maximal rank at each point of R_{Q^2}.

Our aim is to compute $\int_\Sigma \mathrm{d}Vf$, where $Qf = 0$. The fact that the metric g is Q-invariant implies that $\mathrm{div}_{\mathrm{d}V} Q = 0$, which means that the volume element on Σ constructed using the Q-invariant metric is also Q-invariant. Thus, it is easy to see that the following is true:

$$\frac{\partial}{\partial\lambda}\int_\Sigma \mathrm{d}V\cdot f\mathrm{e}^{-\lambda Q\sigma} = -\int_\Sigma \mathrm{d}V\, Q(f\mathrm{e}^{-\lambda Q\sigma}) = 0. \tag{4}$$

Let $\{U_m\}_{m\in I}$ be a finite atlas of Σ and $\{h_m\}_{m\in I}$ a partition of unity on Σ subordinate to this atlas. Suppose also that the atlas is chosen to satisfy the following two conditions:

(i) if $\overline{U_k}\bigcap R_{Q^2} \neq \emptyset$, $k\in I$, then $U_k\bigcap R_{Q^2}\neq\emptyset$;
(ii) if $U_k\bigcap R_{Q^2}\neq\emptyset$ then the critical set of $Q\sigma|_{U_k}$ is just $U_k\bigcap R_{Q^2}$.

Using (4) one can rewrite an expression for the integral of f over Σ as follows:

$$\begin{aligned}\int_\Sigma \mathrm{d}Vf &= \lim_{\lambda\to\infty}\int_\Sigma \mathrm{d}Vf\mathrm{e}^{-\lambda\langle Q^2,Q^2\rangle}\\ &= \sum_{m\in I}\lim_{\lambda\to\infty}\int_{U_m}\mathrm{d}Vh_m f\mathrm{e}^{-\lambda\langle Q^2,Q^2\rangle}.\end{aligned}$$

Let $m(Q^2)$ denote the number part of the vector field Q^2 and $R_{m(Q^2)}\subset\Sigma$ the zero locus of $m(Q^2)$. Let us choose $k\in I : \overline{U_k}\bigcap R_{Q^2}=\emptyset$. Then $\overline{U_k}\bigcap R_{m(Q^2)}=\emptyset$. As a consequence of (i) the number part of $\langle Q^2, Q^2\rangle$ is positive at each point of U_k, so one can find such positive constants c_1 and c_2 that

$$\left|\int_{U_k}\mathrm{d}Vh_k f\mathrm{e}^{-\lambda\langle Q^2,Q^2\rangle}\right| \leqslant c_1\lambda^n\mathrm{e}^{-c_2\lambda}\to 0 \qquad \text{as} \qquad \lambda\to\infty.$$

Thus we conclude that

$$\int_\Sigma \mathrm{d}Vf = \sum_{\{k\in I|U_k\bigcap R_{Q^2}\neq\emptyset\}}\lim_{\lambda\to\infty}\int_{U_k}\mathrm{d}Vh_k f\mathrm{e}^{-\lambda\langle Q^2,Q^2\rangle}. \tag{5}$$

The integrals on the right-hand side of (5) can be calculated using the Laplace method adapted to include integrals over superspaces (see e.g. [6]). Under the condition that the odd codimension of $R_{Q^2}\subset\Sigma$ is equal to its even codimension, we obtain that

$$\lim_{\lambda\to\infty}\int_{U_k}\mathrm{d}Vh_k f\mathrm{e}^{-\lambda\langle Q^2,Q^2\rangle} = \int_{U_k\bigcap R_{Q^2}}\mathrm{d}v_Q(h_k f)|_{U_k\bigcap R_{Q^2}} \tag{6}$$

where $\mathrm{d}v_Q$ is the volume element on $U_k\bigcap R_{Q^2}$ defined as a partition function of degenerate functional $Q\sigma|_{U_k}$ (see [6], lemma 2). By (5) $Q\sigma$ depends on Q and g only, so does $\mathrm{d}v_Q$. It remains to note that the set $\{h_k|_{R_{Q^2}}\}_{\{k\in I|U_k\bigcap R_{Q^2}\neq\emptyset\}}$ provides one with the partition of unity on R_{Q^2}. Therefore, substituting (6) into (5) and using the definition of the integral over a (super)manifold, we see that

$$\int_\Sigma \mathrm{d}V\cdot f = \int_{R_{Q^2}}\mathrm{d}v_Q\cdot f|_{R_{Q^2}}.$$

The theorem is proved. $\square$

Corollary. *If $R_{Q^2} = R_Q$ the theorem implies the localization of corresponding integrals over Σ in the sense of definition (1).*

Note that the statement of the above theorem can be formally justified in the case when Σ is an infinite-dimensional manifold. For example, Σ can be realized as a space of maps from a world sheet to a target manifold of some quantum field theory. This suggests that there are possible applications of the above theorem which are different from those we consider below.

Finally, let us remark that if $R_{Q^2} \neq R_Q$ one can still prove the localization of the integrals under consideration to R_Q. The proof will consist of two steps: first, one repeats the above arguments to prove the localization to R_{Q^2}; second, one notes that the vector field Q generates a nilpotent vector field on R_{Q^2} and $f|_{R_{Q^2}}$ is invariant with respect to this vector field. The corresponding integral is localized to R_Q (see e.g. [8, 12]).

2. Derivation of the main result

Let Σ be a compact supermanifold. Suppose that Q is an odd vector field on Σ. We always assume that the zero locus R_Q of the vector field Q is a submanifold of Σ and that Q is non-degenerate in the neighbourhood of R_Q. Let $\mathrm{d}V$ be a fixed Q-invariant volume element on Σ. Assume that Q satisfies the localization conditions, i.e. (1) holds for any Q-invariant function f on Σ. Let M be another supermanifold. To avoid irrelevant technicalities, we suppose that M is diffeomorphic to a linear superspace. Denote by E the (super)space of maps from Σ to M. Naturally, an action of Q on Σ generates an infinitesimal diffeomorphism of the space of maps:

$$\Phi \to \Phi + \epsilon Q\Phi \tag{7}$$

where $\Phi \in E$ and ϵ is an odd parameter. We will use the notation $\hat{Q}$ for the vector field on E corresponding to (7).

Next let us impose an additional condition on Q which will be crucial for further considerations. Namely, we will assume that the following Cauchy problem has a solution:

$$Q\Phi = 0 \tag{8}$$

$$\Phi_{R_Q} = \Phi_0 \tag{9}$$

where Φ_0 is any map from R_Q to M. In other words, we require that any map $R_Q \to M$ can be continued to the Q-invariant map $\Sigma \to M$. In all interesting cases, the problem (8), (9) has a lot of solutions. We suppose that the space of solutions of (8) corresponding to a fixed initial condition (9) is contractible.

Consider now a quantum field theory defined on Σ. Let $\mathcal{L}(\Phi, \partial\Phi)$, $\Phi \in E$, be a corresponding quantum Lagrangian. The word 'quantum' means that having started from classical field theory we fixed gauge-like symmetries of the classical Lagrangian using some quantization procedure (BV for example, see [1]) and arrived at the expression for $\mathcal{L}(\Phi, \partial\Phi)$ where Φ is a map from Σ to the manifold M of both physical and auxiliary fields such as ghosts, antifields, etc. Therefore the corresponding action functional is non-degenerate, i.e. the linear integral operator in $T_\Phi(E)$ with the kernel $\frac{\delta^2 S}{\delta\Phi^i(x)\Phi^j(y)}$ has no zero eigenvectors for any $\Phi \in E$. Here $x, y \in \Sigma$ and a choice of local coordinates in M is assumed.

The main condition imposed on the quantum field theory at hand is $\hat{Q}$-invariance. Namely, we suppose that

$$\mathcal{L}(\Phi + \epsilon Q\Phi, \partial(\Phi + \epsilon Q\Phi)) = \mathcal{L}(\Phi, \partial\Phi) + \epsilon Q\mathcal{L}(\Phi, \partial\Phi). \tag{10}$$

The fact that the action $S = \int_\Sigma \mathrm{d}V\mathcal{L}$ is $\hat{Q}$-invariant follows then from (10) and the Q-invariance of the volume element $\mathrm{d}V$.

There is a simple construction generating a lot of models satisfying (10). Let h_n be a multivector field of rank n on Σ. Introducing local coordinates $\{z^\alpha\}$ on Σ one can present h_n in the following form:

$$h_n = h_n^{\alpha_1\alpha_2\ldots\alpha_n}(z)\frac{\partial}{\partial z^{\alpha_1}} \otimes \frac{\partial}{\partial z^{\alpha_2}} \otimes \cdots \otimes \frac{\partial}{\partial z^{\alpha_n}}.$$

Using a multivector field h_n and a map $\Phi : \Sigma \to M$ one can construct a map h_{*n} from Σ to the nth tensor power of the tangent bundle TM over M. Choosing local coordinates both in Σ and in M one can present it as follows:

$$\begin{aligned} h_{*n}(z) &= h_n^{\alpha_1\alpha_2\cdots\alpha_n}(p)\frac{\partial\Phi^{i_1}}{\partial z^{\alpha_1}}\frac{\partial\Phi^{i_2}}{\partial z^{\alpha_2}}\cdots\frac{\partial\Phi^{i_n}}{\partial z^{\alpha_2}} \\ &\equiv h_n(\Phi^{i_1} \times \Phi^{i_2} \times \cdots \times \Phi^{i_n})(z). \end{aligned} \tag{11}$$

Suppose now that h_n is Q-invariant, i.e. $L_Q h_n = 0$, where L_Q is a Lie derivative with respect to the vector field Q. Then it is easy to see

$$\epsilon Q(h_n(\Phi \times \cdots \times \Phi)) = h_n((\Phi + \epsilon Q\Phi) \times \cdots \times (\Phi + \epsilon Q\Phi)) - h_n(\Phi \times \cdots \times \Phi). \tag{12}$$

Suppose finally that the derivatives of Φ enter the Lagrangian only in the form of combinations (11), where h_n is a Q-invariant multivector field. Then in virtue of (12) relation (10) is satisfied and the corresponding action functional is $\hat{Q}$-invariant.

Let us illustrate the above considerations with the following example. Take g to be a Q-invariant metric on Σ. Then the following model is $\hat{Q}$-invariant:

$$S = \int_\Sigma \mathrm{d}V\left(g^{\alpha\beta}\partial_\alpha\Phi^i\partial_\beta\Phi^j G_{ij}(\Phi) + V(\Phi)\right). \tag{13}$$

Here $g^{\alpha\beta}$ is a Q-invariant multivector field of rank 2 inverse to the metric tensor $g_{\alpha\beta}$ and G_{ij} is a metric tensor on M. Note that (10) constitutes a natural non-linear generalization of the Parisi–Sourlas model [5].

Now we are able to formulate the main result of the paper. The Q-invariant (Schwinger) correlation functions of the theory described above have the following generating functional:

$$Z[J] = \int [\mathrm{D}\Phi]_E \exp\left(\mathrm{i}\beta\left(S[\Phi] + \int_\Sigma \mathrm{d}V J_i(p)\Phi^i(p)\right)\right) \tag{14}$$

where $\{J_i\}$ are Q-invariant functions on Σ playing the role of sources, $[\mathrm{D}\Phi]_E$ is a formal measure on the space of maps E and β is a coupling constant.

By means of formal manipulations with functional integrals, we are going to show that under the conditions on Q and $S[\Phi]$ the generating functional (14) can be rewritten as follows:

$$Z[J] = \int [\mathrm{D}\Phi]_{E_Q} \exp\left(\mathrm{i}\beta\left(S[\Phi\,|_{R_Q}] + \int_{R_Q} \mathrm{d}v_Q J_i(p)\Phi^i(p)\,|_{R_Q}\right)\right). \tag{15}$$

Here E_Q denotes the space of maps from R_Q to M, $[\mathrm{D}\Phi]_{E_Q}$ is a measure on E_Q; the new action functional is

$$S[\Phi|_{R_Q}] = \int_{R_Q} \mathrm{d}v_Q\mathcal{L}\left(\Phi|_{R_Q}, \partial'\Phi|_{R_Q}, 0\right) \tag{16}$$

where the new Lagrangian is obtained from the old one by restricting the fields to R_Q and setting the derivatives of the fields in the directions transversal to R_Q equal to 0. We also used the symbol ∂' to denote the derivatives along R_Q.

Equation (15) states the equivalence between the Q-invariant sector of the initial theory and the theory determined by the action functional (16) defined on the submanifold of the initial source manifold Σ. According to the adopted terminology, dimensional reduction occurs.

Note that in the case when R_Q is zero dimensional, the rhs of (15) reduces to a finite-dimensional integral, which means an exact solvability of the Q-invariant sector of the theory we have started with. We also see that in the situation when Q happens to have no zeros at all the Q-invariant sector is trivial which yields a set of Ward identities for the correlation functions of the initial theory.

To demonstrate the equality between (14) and (15) let us consider first the subset $\mathcal{R}_Q$ of E consisting of Q-invariant maps from Σ to M. The space $\mathcal{R}_Q$ is foliated by means of the following equivalence relation: two Q-invariant maps $\Phi, \Phi' \in \mathcal{R}_Q$ belong to the same fibre of the foliation iff $\Phi|_{R_Q} = \Phi'|_{R_Q}$; in other words Φ and Φ' are equivalent if they determine the same element of $E_Q = \{R_Q \to M\}$. Consider a section of such foliation—a map $\tilde{\Phi} : E_Q \to \mathcal{R}_Q$ which assigns to each element of E_Q *a unique* element of $\mathcal{R}_Q$. In other words, we set a rule which singles out one and only one solution to the problem (8), (9) for each $\Phi_0 = \Phi_{R_Q}$. Such section exists due to the stated assumptions about the space of solutions of the problem (8), (9). Consider now the following functional on E:

$$F[\Phi] = \int_\Sigma \mathrm{d}V G_{ij}(\Phi) \left(\Phi^i - \tilde{\Phi}(\Phi_0)^i\right)\left(\Phi^j - \tilde{\Phi}(\Phi_0)^j\right). \tag{17}$$

Clearly, $\hat{Q}F[\Phi] = 0$. Using (17) we introduce the following deformation of the generating functional (13):

$$Z_\lambda[J] = \int [\mathrm{D}\Phi]_E \exp\left(\mathrm{i}\beta\left(S[\Phi] + \int_\Sigma \mathrm{d}V J_i\Phi^i + \lambda F[\Phi]\right)\right). \tag{18}$$

Note that $Z[J] = Z_0[J]$. Let us show that $Z_\lambda[J]$ is in fact independent of λ:

$$\frac{\partial}{\partial\lambda}\ln Z_\lambda[J] = \mathrm{i}\beta\int_\Sigma \mathrm{d}V\langle F[\Phi(p)]\rangle_{\lambda,J} \tag{19}$$

where $\langle\,\rangle_{\lambda,J}$ denotes the average with respect to the 'action' functional—an argument of exponent in (18). A correlator $\langle F[\Phi(p)]\rangle_{\lambda,J}$ can be considered as a function on Σ. It follows from (17) that the restriction of this function to R_Q is zero. Moreover, this function is Q-invariant as a consequence of the Q-symmetry of the problem. Really, $Q\langle F[\Phi]\rangle_{\lambda,J} = \langle\hat{Q}F[\Phi]\rangle_{\lambda,J} = 0$. The last equality can be regarded as a Ward identity corresponding to the Q-invariance of the vacuum of the theory at hand. Thus the rhs of (19) is an integral over Σ of a Q-invariant function equal to zero on zero locus R_Q of Q. Therefore, it is equal to 0 in virtue of localization condition (1).

So, $Z_\lambda[J]$ is independent of λ. Thus one can compute the generating function $Z[J]$ as follows:

$$Z[J] = \lim_{\lambda\to\infty} Z_\lambda[J]. \tag{20}$$

One can rewrite the rhs of (18) in the following form:

$$Z[J] = \lim_{\lambda\to\infty}\int[\mathrm{D}\Phi_0]_{E_Q}\int_{\{\Phi_{R_Q}=\Phi_0\}}[\mathrm{D}\Phi]_E \exp\left(\mathrm{i}\beta\left(S[\Phi] + \int_\Sigma \mathrm{d}V J_i\Phi^i + \lambda F[\Phi]\right)\right). \tag{21}$$

In the limit $\lambda\to\infty$ the internal integral in (21) localizes to the critical points of the functional $S[\Phi] + \int_\Sigma \mathrm{d}V + \int_\Sigma \mathrm{d}V J_i\Phi^i + \lambda F[\Phi]$ which is defined on the space of maps having a fixed restriction to R_Q. It follows from the Q-invariance of this functional that one of these critical points is $\Phi = \tilde{\Phi}(\Phi_0)$ (see [7] for a proof in the even case). It can be shown under very general assumptions on $S[\Phi]$ that $\Phi = \tilde{\Phi}(\Phi_0)$ is the only extremum contributing to (21) in the limit $\lambda \to \infty$. The contribution can be calculated using an infinite-dimensional version of the stationary phase method. As a result, we obtain the following answer for the generating functional (14):

$$Z[J] = \int [\mathrm{D}\Phi_0]_{E_Q} \exp\left(-\beta S[\tilde{\Phi}(\Phi_0)] + \int_\Sigma \mathrm{d}V\, J_i \tilde{\Phi}(\Phi_0)^i\right) \tag{22}$$

where we absorbed the determinants which appeared as a result of computation of corresponding Gaussian integrals into the redefinition of the functional measure on E_Q. But now we note that by virtue of (10)

$$Q\mathcal{L}(\tilde{\Phi}(\Phi_0), \partial\tilde{\Phi}(\Phi_0)) = 0$$

therefore the integral $S[\tilde{\Phi}(\Phi_0)] = \int_\Sigma \mathcal{L}(\tilde{\Phi}(\Phi_0), \partial\tilde{\Phi}(\Phi_0))$ localizes to the zero locus of the vector field Q. It also follows from the non-degeneracy of Q in the vicinity of R_Q that $\partial_\perp \tilde{\Phi}(\Phi_0)|_{R_Q} = 0$. This remark together with localization condition (1) permits us to conclude that

$$S[\tilde{\Phi}(\Phi_0)] = \int_{R_Q} \mathrm{d}v_Q \mathcal{L}\,(\Phi_0, \partial'\Phi_0, 0). \tag{23}$$

The same localization arguments work for the source term as we have chosen the functions J to be Q-invariant. Substituting (23) into (21) we arrive at expression (15) for the generating functional of the reduced theory.

The way we established the equality between (14) and (15) is somewhat naive in the sense that the result was achieved by means of formal manipulations with the path integral without addressing the questions of proper renormalization of the loop expansion arriving. Our results only suggest the possibility of the phenomenon considered; an additional analysis is required in each particular case.

Keeping up with the level of generality adopted for the present section, we can discuss the relation between instanton sectors in the original and the reduced theory. Suppose that Φ_0 is an extremum of the action functional $S_{\mathrm{red}}[\Phi]$ of the reduced theory, $\Phi : R_Q \to M$. Let $\tilde{\Phi}(\Phi_0) \in \{\Sigma \to M\}$ be a Q-invariant map such that its restriction to $R_Q \subset \Sigma$ coincides with Φ_0. Then $\tilde{\Phi}$ is an extremum of the action functional $S[\Phi]$ of the original theory. The proof of this statement is based on the $\hat{Q}$-symmetry of $S[\Phi]$ and goes along the same lines as its even counterpart (see [7]). Conversely, any Q-invariant extremum of the original theory produces a solution to the equations of motion of the reduced theory by means of restriction. Moreover, any two Q-invariant extrema $\tilde{\Phi}$ and $\tilde{\Phi}'$ of $S[\Phi]$ give rise to the same extremum of $S_{\mathrm{red}}[\Phi]$ given that their restrictions to R_Q coincide, $\tilde{\Phi}_{R_Q} = \tilde{\Phi}'_{R_Q}$. Note also that $S[\tilde{\Phi}] = S[\tilde{\Phi}']$ in virtue of the assumed localization of integrals over Σ with Q-invariant integrals. Thus, we established a one-to-one correspondence between instantons of the reduced theory and critical submanifolds of E consisting of Q-invariant instantons of the original theory having a given restriction to Q. It follows from above that such BPS-like solutions completely determine the instantons contribution to the Q-invariant sector of the original theory. Really, if we suppose for example that the instantons Φ_q of the reduced theory are isolated and classified by an integer q, then by virtue of equality (15) the instanton contribution to the partition function of the original (Wick rotated) theory is equal to

$$\sum_q \frac{\mathrm{e}^{-\beta S_{\mathrm{red}}[\Phi_q]}}{\sqrt{\det Hess(S_{\mathrm{red}}[\Phi_q])}} \tag{24}$$

and is clearly determined by Q-invariant extrema only.

Our conclusions concerning the dimensional reduction of supersymmetric field theories generalize and provide the geometrical understanding of the results of [2]. The cited paper contains the first non-perturbative proof of the dimensional reduction of the Parisi–Sourlas model and describes the relation between instanton sectors of the Parisi–Sourlas model on a linear (3, 2) space and its reduction which is a bosonic theory in dimension 1.

3. Applications and conclusions

Now we would like to explain the relation of the Parisi–Sourlas model to the above discussion.

Consider a supermanifold $\Sigma = B \times \mathcal{R}^{(2,2)}$, where B is a (super)manifold. Let M be a linear superspace. Let us choose local coordinates $\{x^i, y^\alpha, \theta, \overline{\theta}\}$ on Σ, where $\{x^i\}$ is a set of local coordinates on B, $\{y^\alpha, \theta, \overline{\theta}\}$ are even and odd coordinates on $\mathcal{R}^{(2,2)}$. Let h be a Riemannian metric on the manifold B. Then a metric on Σ can be defined by means of the following quadratic form:

$$g = h_{ij}(x)\delta x^i \delta x^j + \sum_\alpha \delta y^\alpha \delta y^\alpha + 2\delta\theta\delta\overline{\theta}. \tag{25}$$

Consider the following σ-model having Σ as a source manifold:

$$Z[\beta] = \int [\mathrm{d}\Phi]_E \, \mathrm{e}^{\mathrm{i}\beta S[\Phi]} \tag{26}$$

$$S[\Phi] = \int_\Sigma \mathrm{d}V (g^{-1}(\Phi^I, \Phi^J) G_{IJ}(\Phi) + V(\Phi)) \tag{27}$$

where $E = \{\Sigma \to M\}$, $\Phi \in E$; g^{-1} is a bivector field on Σ inverse to the quadratic form (25). In components $\Phi^I = \phi^I + \psi^I\overline{\theta} + \overline{\psi}^I\theta + A^I\theta\overline{\theta}$. It follows from the results of [5] that the model (26), (27) can be viewed as a result of stochastic quantization of a σ-model defined on the space of maps $\{B \times \mathcal{R}^{(2,0)} \to M\}$. The corresponding action functional is

$$S[\phi] = \int_{\Sigma_0} \mathrm{d}V \left(\left(h^{ij} \frac{\partial\phi^I}{\partial x^i} \frac{\partial\phi^J}{\partial x^j} + \sum_\alpha \frac{\partial\phi^I}{\partial y^\alpha} \frac{\partial\phi^J}{\partial y^\alpha} \right) G_{IJ}(\Phi) + V(\phi) \right). \tag{28}$$

In such an interpretation, we consider only the following correlation functions of the model (26), (27): $\left\langle \Phi^{I_1}\big|_B \cdots \Phi^{I_k}\big|_B \right\rangle$. It is easy to check that the metric form (25) is invariant with respect to the following odd vector field on Σ:

$$Q = \overline{\theta} \frac{\partial}{\partial y^1} + \theta \frac{\partial}{\partial y^2} + y^2 \frac{\partial}{\partial \overline{\theta}} - y^1 \frac{\partial}{\partial \theta}. \tag{29}$$

This vector field satisfies all conditions of the corollary and the theorem which yields the integration formula (1) for the integrals of Q-invariant functions over Σ.

Consider the Q-invariant sector of the model (26), (27). This sector describes in particular stochastic correlation functions of the model (28). The results of the previous section suggest that this sector is equivalent to the following model defined on the manifold B:

$$Z[\beta] = \int [\mathrm{d}\phi]_{E_B} \mathrm{e}^{\mathrm{i}S[\phi]} \tag{30}$$

$$S[\phi] = \int_B \mathrm{d}v \left(g^{ij} \partial_i \phi^I \partial^j \phi^J G_{IJ}(\phi) + V(\phi) \right) \tag{31}$$

where $E_B = \{B \to M\}$ and $\phi \in E_B$ and $\mathrm{d}v$ corresponds to the metric h on B. This conclusion agrees with corresponding statements about dimensional reduction of the original Parisi–Sourlas model and its modifications considered in [4].

Let us note that all results of the present section can be generalized to the case when Σ is a total space of a flat (m, m)-bundle over the base B.

We would like to conclude with the following remark. Standard applications of localization techniques to quantum field theory deal with localization of integrals over target spaces. This leads to well-known cases of exactness of semiclassical limits in topological and integrable quantum field theories, see [11] for more information. Our work shows that localization of integrals over world sheets of supersymmetric field theories leads to dimensional reductions of the latter. This allows us to put the Parisi–Sourlas work in the general perspective of the theory of localization.

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